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Pósa-Method & Cubic-Geometry – A Sample of a Problem Thread for Discovery Learning of Mathematics[1]

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The Pósa-method is one of the most prominent examples of current Hungarian practice of inquiry learning of mathematics for gifted upper-primary and secondary school students, highly characterized by the use of web of problem threads. Samples of connected threads are presented, with geometric and combinatorial problems relating to constructions of unit cubes, with their solutions and the analysis of the connections between these problems, the so called kernels of the threads, as principal means of triggering the discovery learning process.

Inquiry-based learning of mathematics in Hungary – the Pósa camps

Discovery learning of mathematics has a long and to some extent influential tradition in Hungary,[2] hallmarked by the theoretical works of György Pólya and Imre Lakatos (Lakatos, 1975; Polya, 1954, 1957) and some exceptional teaching practices, such as the complex mathematics teaching experiment of Tamás Varga in the 1970s (Halmos & Varga,

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[2] Pósa method is referred to as a type of discovery learning of mathematics method, the English translation of the Hungarian term Lajos Pósa uses regularly to describe the set of coherent ideas behind his teaching practice. However the term 'discovery learning' in the international context of research in mathematics education has a highly multiform appearance, and methods described as discovery may show a great diversity of the level and type of similarity to the Pósa method. Another potentially appropriate term is 'inquiry-based learning' and correspondingly 'inquiry-based education', which may have an even wider range of connotations, as well as the problem-solving tradition. The high level of overlapping of the use of these terms, when describing student-centered approaches to mathematics education (Artigue & Blomhøj, 2013) the term preferred by Lajos Pósa is used primarily.

1978). However, with the exception of oeuvres of some prominent teachers or some movements in talent care, the present state of mathematics teaching in Hungary does not seem to dominantly reflect ideas of promoting individual thinking and discovery, neither the importance of the enjoyment of doing mathematics, as if we forgot what we should have always been proud (and use) of. Nevertheless, the counter-examples, the relatively few those who believe in inquiry-based type of learning have been developing promising and exceedingly elaborate methods.

One of the most notable figures among these pioneers is Lajos Pósa, a mathematician and a mathematics teacher, has been organizing and leading his own weekend math camps for talented 7^{th} – 12^{th} grader students in Hungary since 1988, during which period he and his disciples have been developing his own discovery teaching of mathematics method. For details about being a member of a camp, the invitation procedure and organizational issues please consult the website of The Joy of Thinking Foundation (http://agondolkodasorome.hu/en/).

For more than a decade, former students of Pósa, who are now researchers, teachers or applied professionals of mostly mathematics related fields of science, the grown-up disciples of Pósa Lajos, have also been conducting their own weekend camps, using the same method, and mostly the same, but continuously changing and broadening content material. The second author of the present paper is one these camp leaders, as well as Péter Juhász, who is the leading organizer of the annual Camp of Mathematical Amusements. For short description of the camp, abbreviated as MaMuT in Hungarian, look at (Juhász, 2012:50-51) or (http://agondolkodasorome.hu/mamut/) for an updated, though Hungarian version.

In the framework of the Content Pedagogy Research Program of the Hungarian Academy of Sciences new experimental programs have been launched, such as the Flying School, in order for broadening the set of target students, to modify the method to be applied with less talented students as well, and spreading the it around the country.

Principles of the Pósa method and main characteristics of its teaching practice

The main goal of the Pósa method type of teaching assisted mathematics sessions is to offer the opportunity for students to be happy by thinking on interesting mathematical problems and discovering mathematical connections. Therefore the two main principles are *students' enjoyment* of doing mathematics and promote *individual* (or group) *thinking*, discovery learning of mathematics, "children should be taught how to think, rather than making them learn theorems and formulas by heart, or giving them ready-made methods to solve problems" (Juhász, 2012:51).

Letting *students pose questions* (new problems, connected to the ones under study), and teaching them how to do this, as well as *integrating their questions* into the program of study are also important factors, students take an active part in planning the learning process. The *freedom of making mistakes* is indispensable for an inquiry-based type of learning process, students should not be afraid of making errors.

The type and structure of interactions between all participants are of high importance. Student-to-student interactions, *group work* is typical and encouraged, but it may have a slightly less important role compared to the aforementioned principles. *Student-teacher dialogues* are well-designed, serving the purpose of letting children think as much as possible, but not leaving them get stuck. The study of the characteristics of student-teacher dialogues in the Pósa method is a topic of further studies.

The function of the context of the problems, the actual wording of them, or the *role of tales* (as teachers of the method usually call them) or word problems also forms a crucial part of the method. On top of that, one of the most decisive and distinctive characteristic of the method is the use of *web of problem threads.*

Web of problem threads triggering the discovery learning process[3]

In this paper the focus is on the analysis of problems, and their use in problem-solving processes, not mainly as isolated items, but as elements of a coherent whole, with a main focus on the relationships between them. The issue of analyzing problems series in mathematics education, with a focus on the principles of ordering the problems is a current research topic in the fields of the didactics of mathematics (Gosztonyi, 2016) and in the history of sciences, where problems are treated in a broader sense and from a strongly historical point of view (Cifoletti, 2015). A study of the application of the research results in teacher education can be found in (Bernard & Gosztonyi, 2015). The structure of problem series, and the principles behind their structuration appearing in Tamás Varga's work and reform movement in the 1970s and studied in (Gosztonyi, 2016), also the problem series in (Péter, 1961), just to mention some major examples from the Hungarian tradition may show characteristic common features with Pósa's web of problem threads, which is a topic of future research to be conducted by the authors. However, the notion of problem series, reflecting a linear type of ordering, seems not to be perfectly adequate to describe the structure of the web of problem threads used in the Pósa method.

[3] The term 'thread' in connection with problem-solving is being introduced by the authors in the present paper, and originates from Péter Juhász, so far unpublished.

Problems of one thread are connected to each other, which can be realized in different ways. One aspect of this connection is that the problems may be, to differing extent, built on each other, meaning that the anticipated or regular solution to one problem (problem A) needs certain ideas, methods, types of mental representations, etc., or in general, any separable element of thinking, that can be made more easily available for students by thinking on and solving another problem (problem B), or even some other problems. This link between problem A and problem B also presents itself as form of a common feature, or common element, called the kernel of the thread, [4] which is another aspect of the connection.

These threads are not isolated, they may have meeting points, common problems, forming a web of problem threads, and during the mathematics sessions they run simultaneously, that is (usually) problems from at least two threads are presented to students at the same time. The detailed research on the structure of this kind of web of problems and their use in the Pósa method has just started, there are still many open questions and yet vague concepts.

Regarding the content of the problems, they show a great diversity. In the present paper, the focus is laid on geometry, problems with constructions made from cubes. However, behind the apparently solely geometrical problems, many other areas of mathematics appear to present themselves, such as combinatorics and (basic) number theory.

A sample from a web of problem threads on cubic geometry, with solutions and notes

In the followings a sample from a web of problems is presented and analyzed, parts of three threads that are connected within the web. These problems, forming the whole web, were constructed by the second author, a former student and now a colleague of Lajos Pósa, one of the teachers of the Pósa camps, and it was published in a special methodological issue for mathematics teachers (Szűcs, 2016).

Although we use the sole word 'problem' for all the tasks to be analyzed, in the aforementioned issue they appear under two different names, translated here as challenges and exercises. The main difference, in the whole web of problems, is that the main purpose of the exercises is to make the thinking process more colorful, to add some interesting element to the discussed topic, but they are usually not needed for solving the subsequent problems. However there exceptions, where the exercises can also play an important role in the building process, as in the following sample. The original numeration of the problems, as they appear in (Szűcs, 2016), is presented in parentheses, while the ones that denotes their place in some main threads appear in braces.

[4] The term „kernel of a thread" is being introduced by the authors in the present paper.

Problem 1 (Challenge 1) {A1}

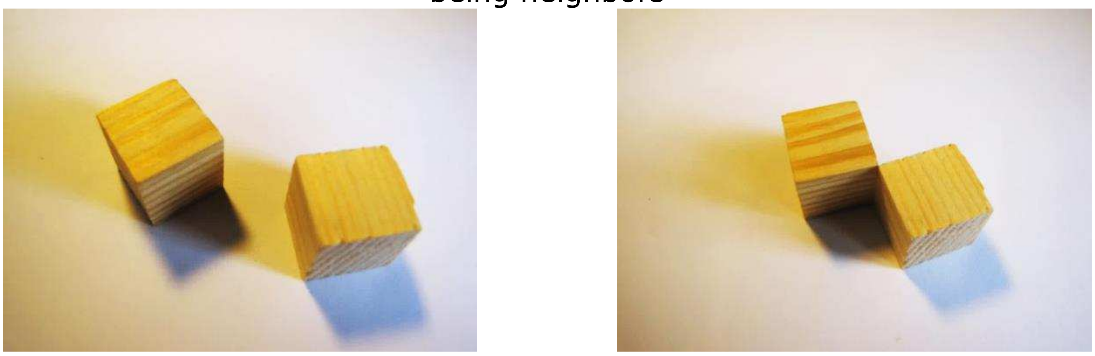
In the following problems we work with constructions, buildings made of little wooden cubes, so called unit cubes. We call two cubes neighbors if and only if they are joined completely or just partly by their faces.[5] The buildings need to be connected.[6]

/a) Place 8 unit cubes in a way that each has an odd number of neighbors.

/b) Try to make it with an even number of neighbors for each unit cube.

Figure 1. Two cubes that are neighbors. Partially touching on the left, and full-face touching on the right

Figure 2. Two cubes which are not neighbors. Edge touching is not enough for being neighbors



[5] In other words, two cubes are neighbors if and only if they either touch completely along a common face, or just parts of their faces are common, as demonstrated below on Picture 1. Picture 2 presents two cubes which are not neighbors.

[6] Although we suppose that students intuitively know what 'connected' means, if necessary, it can be explained to them by presenting examples of connected and unconnected constructions, or through definitions, such as the following: there exist a continuous curve between every two points of the construction, all the points of which belongs to the construction.

Solution to Problem 1. The required construction does exist. There are many different good solutions; six of them can be seen in *Figures 3-5.* below.

Figure 3. Two solutions for Problem 1 (1st and 2nd)

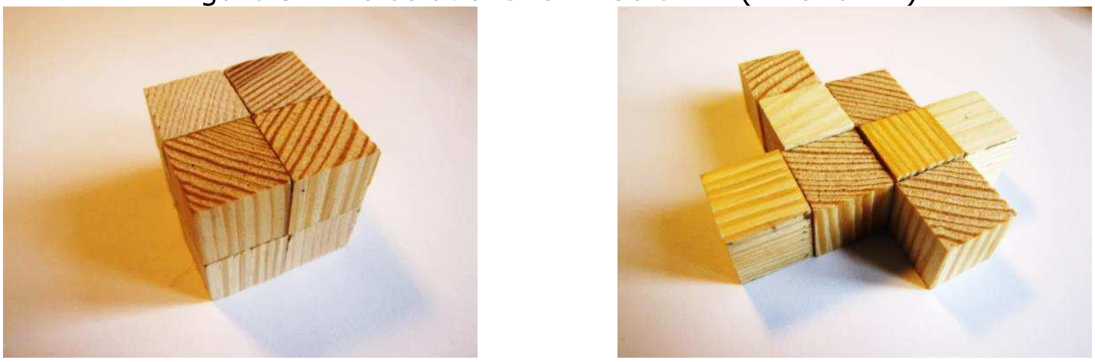


Figure 4. Two more solutions for Problem 1 (3rd and 4th)

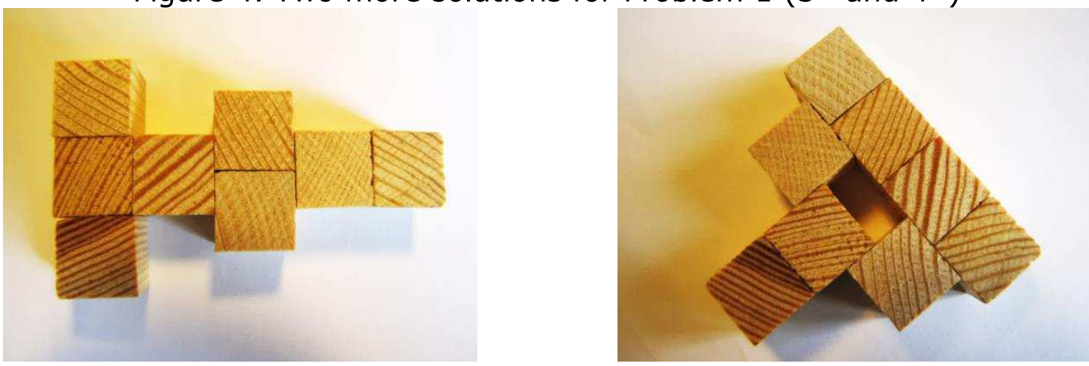
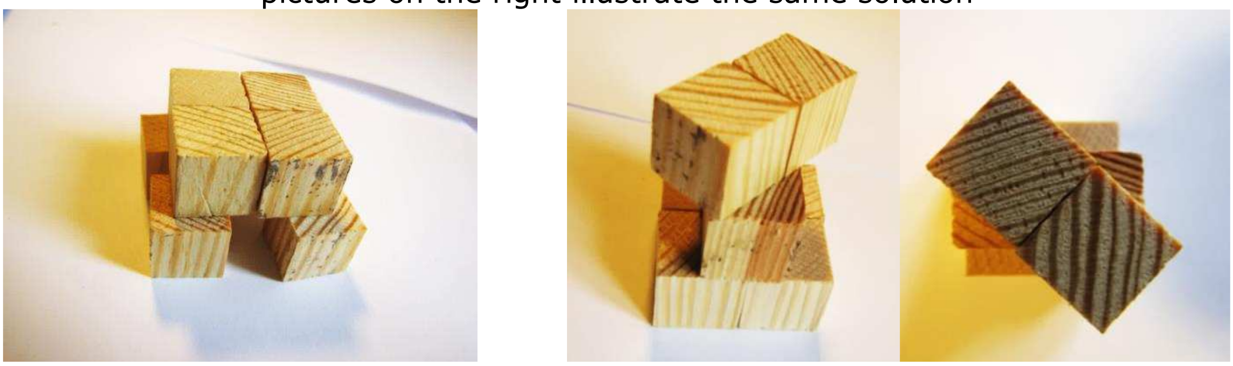


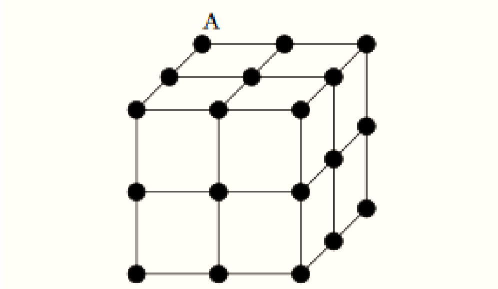
Figure 5. Another two more solutions for Problem 1 (5th and 6th). The two pictures on the right illustrate the same solution



Problem 2 (Challenge 6) {C1}

We are now in the not too distant future, humanity has already colonized space. The protagonist of our story is an astronaut who lives in a space station that consists of 27 space modules. The modules are set at the vertices of the little unit cubes that make a $2 \times 2 \times 2$ cube. There are passages between each neighboring modules, represented by the edges of the unit cubes. Our astronaut can only use these passages to move between the modules.

Figure 6. The space station



Our hero is now at the module signed by vertex A, in Figure 6, and would like to go to the opposite vertex of the two-unit cube. In how many different ways can they do this,[7] if they do not want to move away from their target?

Solution to Problem 2. The different ways the astronaut can move along are different series of connected passages along which they do not move away from the target.[8] We may suppose that the astronaut starts from vertex A.[9] Our first observation is that they can move only forwards, to the right or downwards,[10] if moving along the edges, otherwise they would move away from their target. After that, we count the number of different paths to hither vertices. Obviously there is only one appropriate path to a neighboring vertex, a one edge long one. If we would like to count the number of different paths to such a vertex the number of possible paths to whose possible vertices of arrival is already known,[11] then we can calculate it by taking the sum of the numbers of paths to the possible vertices of arrival of the given vertex. Therefore, the right solution can be calculated on the basis of the numbers of Figure 7, the astronaut can choose among 90 different appropriate paths.

[7] The reason for the choice of using the epicene (gender-neutral) pronoun, 'they' in the English translation of the text is the following. In the original Hungarian text, when the corresponding gender-neutral word for astronaut is only covertly present, the reference is carried out by not the use of a pronoun; it is expressed by epicene suffixes. Therefore, throughout the whole web of problems, any person appearing in the stories are expressed by epicene grammatical means. Therefore, in the English translation, instead of the more natural use of the gender-specific pronouns 'he' and 'she', the epicene singular pronoun 'they' is used. The gender-neutral characteristic of the language use in the text is a mean of decreasing the potential difference in self-confidence related to mathematical activities between girls and boys.

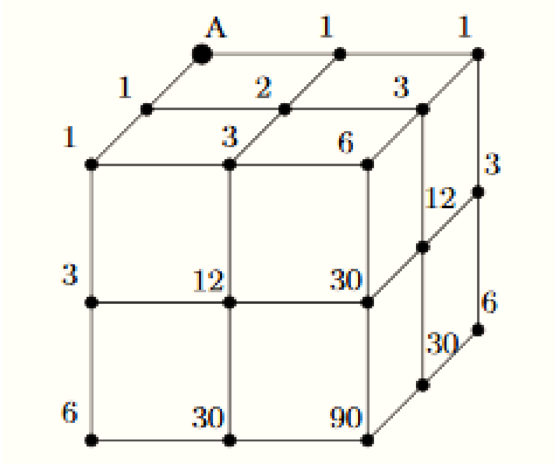
[8] Two passages, that are edges are connected (in a path) iff they connect one module, that is, they have a common vertex, and the path goes through this common vertex.

[9] without loss of generality

[10] The names of the directions are relative, but can (and should be) made obvious by agreeing on their meaning in a given discussion with students.

[11] The 'possible vertex of arrival of a given vertex A', is a vertex B iff there is a passage from vertex B to vertex A, and having this passage in the path fulfills the requirement that the astronaut does not move away from their target.

Figure 7*.* The numbers of different paths of the astronaut from vertex A



Problem 3 (Challenge 7) {A2}

Now 2 unit cubes are called neighbors iff they are joined by their whole faces,[12] and we place all cubes in a paraxial position.[13]
The buildings still need to be connected.
/a) Place 16 cubes in a way that each has an odd number of neighbors.
/b) Try to make it in a way that you have as many cubes at the second floor as you can.

Solution to Problem 3/a. The required building can be constructed in several different ways, *Figures 8-12.* illustrates some of them. The buildings are represented by pictures of the real constructions, as well as by schematic top-view representations in the form of charts in which the numbers denote the number of unit cubes placed on top of each other at the corresponding 'vertical column'.

Figure 8. Recursive solution to Problem 3. The majority of the solutions of students resemble to this one

[12]'iff' meaning 'if and only if', as frequently used in mathematical texts.
[13] The cubes are in a paraxial position, if any two faces of the cubes in the construction are either parallel or perpendicular

Figure 9. Another recursive solution to Problem 3/a. It starts from 2-unit cube solution to Problem 1. It is an axially symmetric solution. There are 5 cubes on the 'second floor', which makes it quite a good solution for Problem 3/b

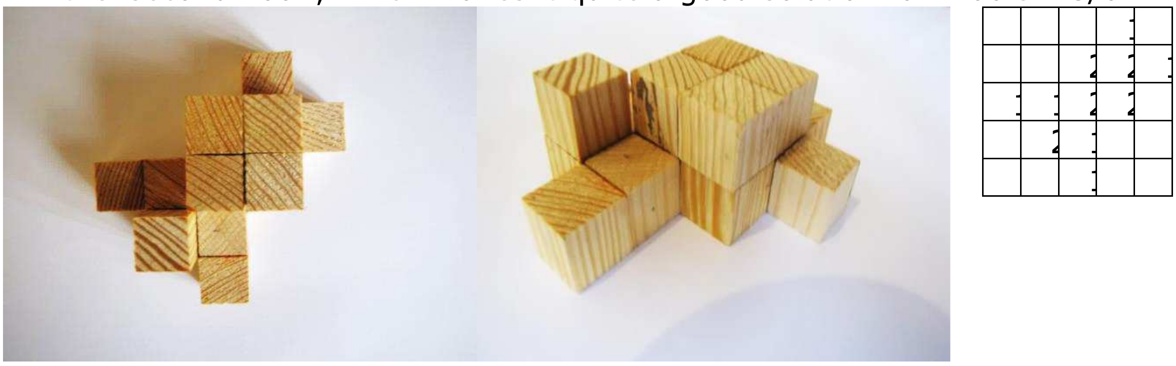


Figure 10. Another axially symmetric solution to Problem 3/a. However there are less cubes on the 'second floor', that is a relevant issue in Problem 3/b

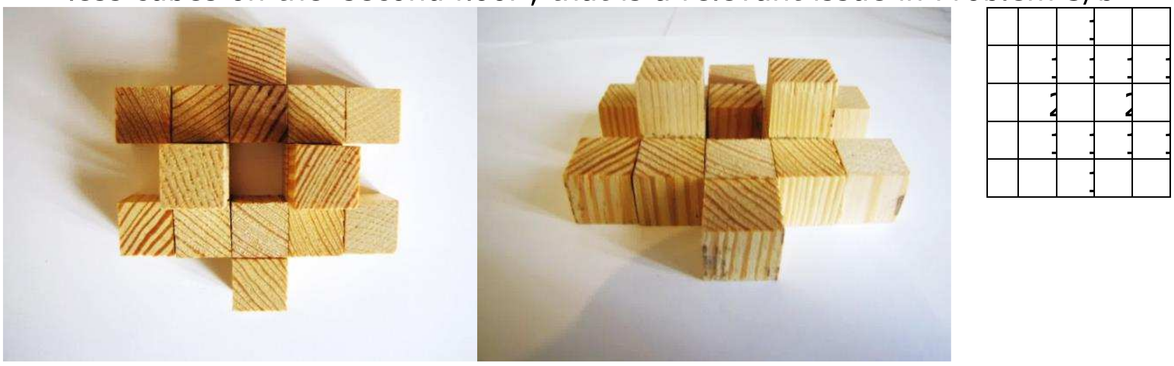
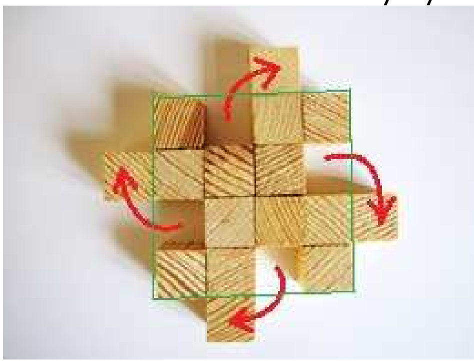
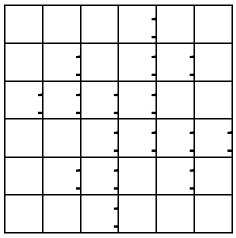


Figure 11. 'Single floor' solution to Problem 3/a. It is also an axially symmetric solution, and it can also be considered a recursive solution as well. A generalization of this construction makes it applicable for any even number of unit cubes each having an odd (1 or 3) number of neighbors

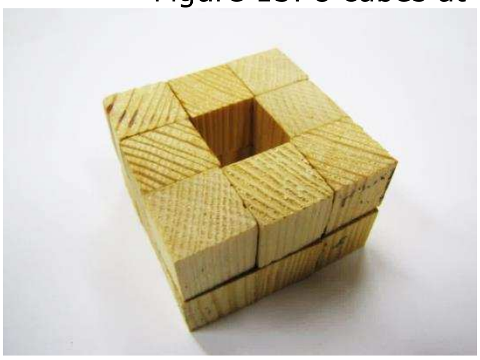
Figure 12. A 'false position-like' solution to Problem 3/a.[14] It is also a 'single floor' and a centrally symmetric solution

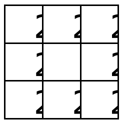




Solution to Problem 3/b. As every unit cube at the second floor has to be on the top of a unit cube at the first floor, there have to be at least as many cubes at the first floor as at the second one. Therefore, you cannot place more cube to the second floor than 8 pieces, which can be constructed as shown by Figure 13 below.

Figure 13. 8 cubes at the second floor





Constructions with less than 8 cubes at the second floor can also be of great value and a source of student success, worth to be presented to mates as well.

Problem 4 (Challenge 8) {A3}

Construct a building from as few cubes as you can, that each three projection of it is a 3×3 square. [15] (Cubes need to be in a paraxial position.)
b)[16] The astronauts of the International Space Station also like this problem and they started to think about it. However, it is easier for them,

[14] For notes on the 'false position-like' method see the latter part of the paper, the section called 'Kernels of threads - connection between the problems of the discovery learning process'.
[15] paraxial orthographic planar projections, that is like the „view" of the building from particular „sides"
[16] Obviously, the previous part of Problem 4 could be referred to as Problem 4/a, and is treated as Problem 4/a, although it is not written in the original text.

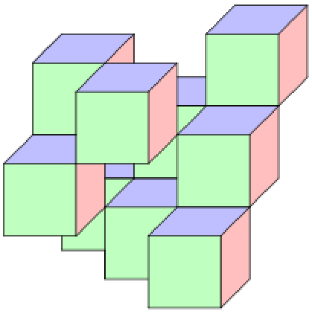
as their cubes can float.[17]
With how many cubes can they solve the problem?

Solution to Problem 4/a.

Figure 14. all the three projections are whole 3×3 squares

Solution to Problem 4/b.

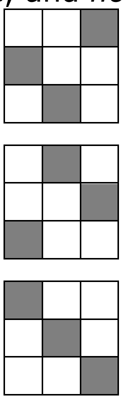
Figure 15. all the three projections are whole 3×3 squares, *spatial* on the left, *by layers* (floors) in the middle, and *floor map* on the right



layer 1:

layer 2:

layer 3:



1st	2nd	3rd
3rd	1st	2nd
2nd	3rd	1st

Problem 5 (Exercise 9) {C2}

We know that in Challenge 6 the astronaut can choose among 90 different paths. Which one is the shortest?

Solution to Problem 5. Along any appropriate path, the astronaut altogether takes two 'unit steps' (crosses two passages) downwards, two ones to the right and two ones forwards. In the case of any 'unit step', they choose one among the six aforementioned possibilities of 'unit steps'. Therefore, independently of the choice of the particular appropriate path, they take altogether 6 'unit steps' to reach their target.

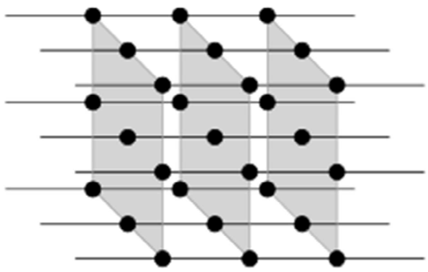
[17] Therefore we treat polycubes here in their general (and original) sense, like in (Masalaski, 1977).

Problem 6 (Challenge 18) {A4}

At the Funfair of Altdorf, the most popular game of the William Tell Cup is a special shooting-gallery. You need to hit the middle of 27 apples placed in the form of a $3 \times 3 \times 3$ cube, with the least number of bowshots possible. Among many successful trials, so far the best was 15 bowshots. The world's best archer is also going to take up the challenge and break the record. [18] They ask as for help, and to make a plan with the least shots possible. (The curse of each shot is a straight line.)

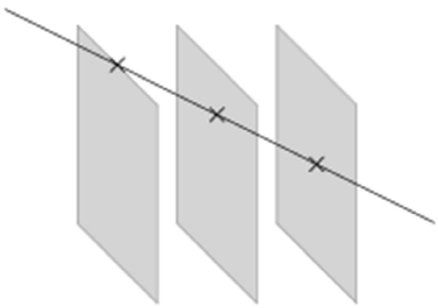
Solution to Problem 6. Let us reformulate the problem by leaving the world of tale. [19] The question is the minimum number of straight lines (bowshots) that contain all the 27 points (the middle of the apples) that are vertices of 8 unit cubes forming a $2 \times 2 \times 2$ cube.[20] It is easy to find a solution with 9 lines, as illustrated by Figure 16.

Figure 16. 9 shots are enough.



With less than 9 shots it impossible to hit all, even for the world's best archer. We show that each straight line goes through 3 or less points, from which the previous statement follows. For each line there is a layering of the cube in a way that the line is not parallel to the layers, see Figure 17. The line can cross each of these 3 layers at most once, therefore it can cross at most 3 points.

Figure 17. Each shot can hit at most 3 apples



[18] The word 'bowman', though would fit better to the context, would not meet the requirement of using gender-neutral forms of language, see Footnote 4.
[19] We do horizontal mathematization in didactic terms. (Treffers, 1987:71)
[20] The midpoints of the unit cubes forming the $3 \times 3 \times 3$ cube appearing in the text of Problem 6 are the vertices of the unit cubes forming a $2 \times 2 \times 2$ cube presented in the solution.

Problem 7 (Challenge 19) {B1}

A contemporary painter has terrible financial circumstances. They are nearly to put in jail, being threatened by 10 of their creditors. In their state of a complete loss of hope, they decide to give their masterpiece, the most brilliant piece of art they have aver created to their creditors, dividing the 5 meters high and 9 meters wide 'The Big Black Rectangle' among the 10 savage creditors. They partition the whole picture into 10 rectangles with integer sides, creating the series called "The Little Black Rectangles". They wish the new pictures to be unique (and therefore more valuable); no two of them can be of the same shape.[21]. Can they do this?

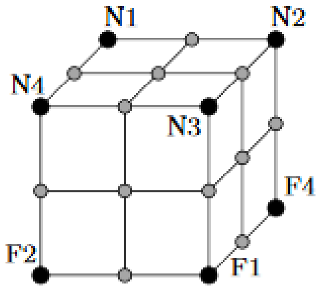
Solution to Problem 7. Unfortunately, they cannot implement the plan. A 10 smallest[22] rectangle with integer sides are the followings:
1×1, 1×2, 1×3, 1×4, 2×2, 1×5, 1×6, 2×3, 1×7, 1×8 or 2×4.
The sum of the areas of these rectangles is: 1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 7 + 8 = 46, that is greater than 5×9 = 45, and therefore we cannot do what was required.

Problem 8 (Challenge 20) {A5 = B2 = C3}

We are in the space again, where 7 new astronauts have arrived at the space station, so now the crew consists of 4 man and 4 women. They live in the space modules illustrated by the vertices in Figure 18, but the passages are all closed due to security reasons. During the spacewalks love springs among two astronauts living at modules which appear as opposite vertices in the $2\times2\times2$ cubic plan of the space station. Therefore, by opening some passages, a corridor is created between the two sweeties, authorized by SOSA (Slovakian Organization of Space Activities. After some months, love springs again, now among other two 'opposite astronauts', and again the third corridor between opposite vertices is opened. In order for avoid the necessity of questions such as "Even in the spring of love thy *love-springs* rot?" (Shakespeare, 1996), the passages should be opened in a way that no two corridors go through the same module. Can it be done with 3 couples? Can it be done with all the 4 couples as well?

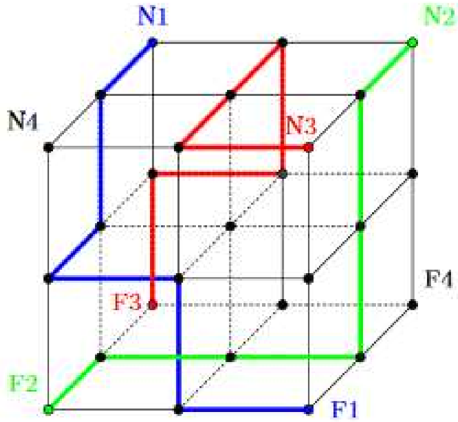
[21] No two of them can be congruent to each other.
[22] the appropriate rectangles with the 10 smallest area

Figure 18. Sweetie astronauts and their modules: the Fellows' (men) and the Non-fellows' (women) modules



Solution to Problem 8. It is quite easy to do it with 2 couples, and it is substantially more complicated, but still available with 3, as one solution is illustrated by Figure 19.

Figure 19. Non-intersecting corridors of 3 couples



There is no solution with 4 couples. Suppose, indirectly, that there exist 4 corridors between the 4 pairs of opposite modules, without intersection (the use of a common module). Based on the result of Problem 5, any path goes through at least 7 modules, and so the appropriate corridors too. As no two corridors can go through the same module, for 4 appropriate corridors we need at least $4 \times 7 = 28$ modules, one more than we have all together. It is impossible, so the indirect supposition cannot be true.

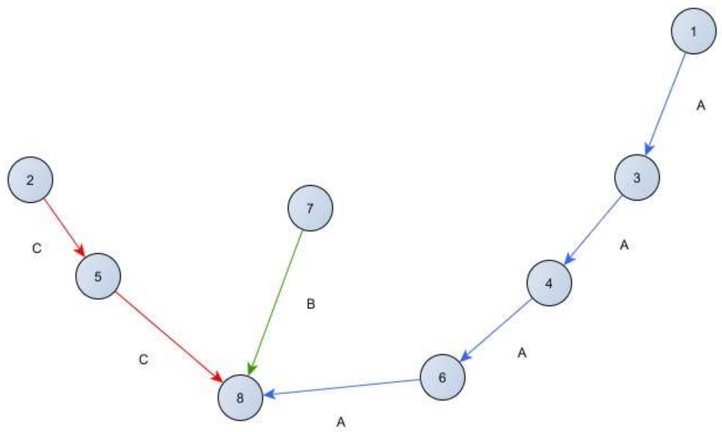
Kernels of threads - connection between the problems of the discovery learning process

Before analyzing the connections between the individual problems, to reveal how the solution of one problem facilitates that of another, a so far not overtly stated principle of the Pósa method and characteristic of the web of problem threads need to be formulated, which formulates the requirement that a problem on one hand, need to be a challenge, at least partly new to the student, and on the other hand, it should not be completely new, it shall share some common features of previous problems, the ones that helps the student to solve it, as the aforementioned facilitating process. All (or most) of problems have to be an *achievable challenge* for students. Besides some constructions, the implementation of which need prominent changes from the usual tracks in mind, the need for proofs of nonexistence and indirect proofs are also characteristic of the sample of problems under examination.

Continuous parts of three different threads of a web are analyzed in brief, one longer and two shorter sections, which are also connected within the web by one distinct problem. The notations thread A, thread B, and thread C are used. It is important to highlight, that the presented parts of the threads are indeed fragments, especially in the cases of thread B and thread C, as only two subsequent problems of a thread would never form a complete thread, threads are typically much longer. It is true for all the analyzed sections of threads, that preceding and succeeding problems usually appear when applied in math camps. In addition, the problems of the analyzed sample are also members of other threads of the whole web of problems in (Szűcs, 2016), which are not revealed in the present paper. In general, updating the process of threading and webbing the problems, adjusting it to the special needs, the interests, talent and previous experience of the students, is typical before and even regularly during each math camp. Therefore the structure of the sample web presented in the paper is only one of many possible and actually implemented variations. It means that the notion 'web of problem threads refer to a rather dynamic kind of entity. It also means that any thread is possibly a fragment of other (though almost the same) threads used or to be used later.

In Figure 20, the graph of the sample is presented, where the numbered vertices refer to the corresponding problems, and the arrows correspond to the examined connections between them. The three directed paths stand for the three thread fragments.

Figure 20. The graph of the sample of the web of problem threads



Thread A (fragment). The supposed kernel of thread A is the change in the geometric, and more and more abstract representation of the problems that students need to use in order for finding the solutions, corresponding to a change in their mental representation of the problem. Problem 1 serves as the first experience with constructions of these unit cubes, allowing the students to get acquainted with the basic properties of these kinds of buildings, and for giving them time to understand the basic concepts used later.

While in Problem 1, it is not so difficult for students to 'see' the whole construction with the requested properties, in Problem 3, the number of unit cubes are too much for the students to be able to think globally (about the whole building instantly) without much effort. They have to develop strategies, during which they think locally, step by step, such as the 'false position-like' solution, or the recursive methods, in order to be successful, that are supposed to be at a more abstract level of thinking. The solution for Problem 3/a, represented by *Figure 12.* is called a 'false position-like' solution, as it can be the result of first building a $4 \times 4 \times 1$ cuboid using the 16 unit cubes, and then replacing 4 pieces to meet the requirement of each having an odd number of neighbors. Therefore, the basic idea behind it is to start from construction that on the one hand, does not meet every requirement, that is a 'false position', but on the other hand, can be built quite easily, and meets some requirements, namely it is connected and made of 16 unit cubes, and then modify this initial construction in some steps to reach the final goal. In this paper we call methods of this kind 'false position-like' methods. The 'false position method' has been used in mathematics for centuries, for instance in ancient Egypt and Mesopotamia, it has had many variations, like the 'double false position method', or a similar method in ancient China called 'the rule of too much and not enough', as described in (Schwartz, 2004).

Dealing with both Problem 1 and Problem 3 students can physically build the constructions using the unit cubes. In Problem 4/b they cannot build it anymore (with the tool presented), they need some other types of representation, like illustrating the individual layers, or the constructing the floor map. The need for these new kinds of representation is particularly important when they present their solutions to each other, and they need to draw on the board. In the solution of Problem 4/b, the need for thinking in layers of the cube is the different kind of representation that appears, which connects it to the next problem of thread A, to Problem 6. Both in Problem 6 and Problem 8, a vertex-edge representation of the cube is needed, having been left the world of solid cubes.

In Problem 3, the unit cubes need to be joined by complete faces, therefore, by solving or trying to solve this problem, students construct polycubes.[23] Problems or puzzles with polycubes have been studied for a long time, for instance in the paper of W. J. Masalski from 1977, where

[23] In Problem 3 only 16-polycubes appear, which are made of 16 pieces of unit cubes. In the subsequent problems n-polycubes with a different n appear as well.

the definition of polycube is given: "A polycube is a solid from cubes joined completely, face to face" (Masalski, 1977:47). However, in Masalski's paper the problems are quite of a different type. One type is counting and building all n-plolycubes for a given n, with n being equal to 2, 3, 4 and 5. The second type is about constructing a large three-unit cube (a $3 \times 3 \times 3$ cube) from given pieces of polycubes. Working on Masalski's problem set, the main emphasis is not on the connections between the individual problems, with the exception of finding the number of different pentacubes, where the author suggests to build upon the solution to the problem of counting the number of different tetracubes: "One systematic way of solving this problem is to take each tetracube and determine how many basically different way a fifth cube can be attached to it" (Masalski, 1977:49). In this respect, the problem set of Masalski's paper seems to be highly different from the problem web studied in the present paper.

Thread B (fragment). Thread B has at least double kernels. First, both solutions are examples of the indirect proof of non-existence. Secondly, the solutions of both problems require finding lower boundaries. Both are to be considered important tools in thinking while doing mathematics, according to the authors. A deep analysis of these kernels, with longer threads, forms the scope of further studies.

Thread C (fragment). Thread C is a rather straightforward example of problems building on each other, as they even either directly refer to each other, as Problem 5 to Problem 2, or there is need to use the result of one problem in solving the other, as the solution of Problem 8 needs the result of Problem 5.

Probably more and perhaps even deeper connections can be revealed between the analyzed problems, which also form part of further research.

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