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High School Students' and Pre-Service Teachers' Drawings of Some Mathematical Concepts

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Mathematical representations help us to think and communicate in mathematical language. Visualization and visual representations are used in mathematics classrooms in order to help us to explore and interpret the meanings of mathematical concepts, relations and procedures. The term visualization represents the spontaneous recognition of mathematical relations. With the term visual representations we refer to all kinds of graphic representations used in mathematics education (drawings, pictures, tables, diagrams, graphs…). Visual representations help students to visualize mathematical concepts and deepen their understanding of those concepts.

The purpose of the study was examining the students' mathematical to understanding for developing instruction to enhance their mathematical knowledge. We aimed to better understanding of learners' mathematical understanding with the help of drawings. Drawings can give us the insight in the type of knowledge that is predominating element in the learners' cognitive scheme. Participants (N=345) were high school students and pre-service teachers (future mathematics teachers, primary pre-service teacher in their 1st year, primary pre-service teacher in their 4th year). Main goal of the study was to find out whether students' representations represent the required mathematical concept.

The results show that participants in our study provided the most adequate representation for subtraction. On the other hand, the least adequate representation was found for calculating the part of a whole. Representation was considered as adequate if drawing could be interpreted in terms of mathematical relations characteristic for given concepts. Subtraction could be recognized in more than 96% of drawings. The result is expected since subtraction is one of the early mathematical concepts. Drawings depicting numerical expressions with parenthesis and exponentiation were found to be adequate in approximately 90%. Calculating the parts of the whole proved to be the most difficult concept to represent with drawings. Less than 75% of participants were able to draw a mathematically sound representation. The result is in accordance with several research findings regarding learners' difficulties with fractions.

Introduction

Drawings are a good indicator for the mathematical knowledge of students (MacDonald, 2013). They are independent mathematical communication tool. The importance of mathematical drawings is the most featured at geometry, where we can actually figure out the way students think (Thom & MacGarvey, 2015). Drawings are therefore often understood as a useful tool in geometry.

Drawings, diagrams and tables represent graphical representations, which are frequently used in mathematics. Some researchers report positive effects of increasing use of graphical representations in mathematics teaching. On the other side, some researchers didn't detect any positive effects in the use of graphical representations. Some of them are even warning about negative effects of pictorial representation oriented instructions (de Bock, van Dooren, Janssens & Verschaffel, 2007).

Based on these findings, some researchers (Güler & Çiltaş, 2011) urge teachers to increase the application of pictorial materials at mathematics teaching. On the contrary, some researchers (Presmeg, 1992) argue that concrete mental pictures may even create a diversion from the essential mathematical relationships. Concrete mental representation can discourage students from important relations, which are needed for effective solving of mathematical problems. Visual representations can also discourage students from the real problem.

Some studies also reported that there are differences in the effectiveness of visual representations for students of different ages. English (1993) claims that tree diagrams for solving combinatorics can be useless for younger students, but they can present a useful and helpful mathematical tool for older students (Beitzel, Staley & Dubois, 2011). In some other studies, gender differences were detected. In these studies, boys performed better (Lowrie & Diezman, 2011).

In some studies (e. g. Badillo et al., 2015), students needed to solve a certain problem. In some cases, students spontaneously decided to solve the problem with the help of drawings. There is no general rule whether students should use the drawings or not. It depends on the type of a task. Tasks for which the students thought process is clearly indicated through a description, explanation, justification and representations that are created when they cope with the task and present their result, Doerr (2006) named modell-ellicting tasks.

Previous studies, which included mathematical graphical representations, were focused on different areas of mathematics. From the area of arithmetic, we can find studies that are conducted on younger students, especially from about developing early number concepts and early arithmetic operations (Rivera, 2014; Badillo, Font & Edo, 2015). In some other studies, graphic representations were given to the students, or students made their own decision if they are going to use the drawings for problem solving tasks. In our research, we wanted students to independently present mathematical concepts with the aid of pictures.

The purpose of the reported study was to examine the participants' mathematical knowledge through analysis of self-generated drawings prompted by symbolic number expressions.

Methods

Participants

The participants in our study were 345 high school students and pre-service teachers. There were 130 high school students, 73 primary pre-service teachers in their 1st year, 96 primary pre-service teachers in their 4th year and 48 future mathematics teachers.

Instruments and procedures

The data were collected through anonymous questionnaire which included 4 different elementary arithmetical mathematical tasks:

- subtraction: $17 - 9$,
- parentheses: $3 \cdot (4 + 5)$,
- fraction: $\frac{3}{5}\ of\ 15$,
- exponentiation: 2^3.

The participants were only instructed to draw a picture which represents different mathematical tasks described above. They didn't get any other instructions. They had 20 minutes to finish this task.

Data analysis

We used an approach that included both qualitative and quantitative research methods. During qualitative content analysis we looked for specific codes through a systematic process of coding. These codes were then combined into categories and the categories were linked to themes. During the coding process objectivity was achieved through employing multiple independent coders. Through the process of coding, precise criteria for categorising participants' drawings into a certain code were defined. In order to strengthen the validity and reliability of the findings that emerged during data analysis (Hesse-Biber & Leavy, 2004). After coding, we used a quantitative methodology and presented data using descriptive and inferential statistics.

Results and interpretation

Our first aim was to find out whether the participants represented adequate mathematical concepts. Table 1 presents the data for each group of participants separately. At the bottom of the table, we present the data for all participants together.

Table 1. Students' representations in terms of adequacy of the represented concept regarding required concept

Type of participants	Type of mathematical operation	Adequate concept		Non-adequate concept		Other		Total	
		f	f%	f	f%	f	f%	f	f%
High school students	Subtraction	124	95,4	0	0	6	4,6	130	100
	Parentheses	114	87,7	0	0	16	12,3	130	100
	Fraction	92	70,8	27	20,8	11	8,5	130	100
	Exponentiation	108	83,1	12	9,2	10	7,7	130	100
	Total	438	84,2	39	7,5	43	8,3	520	100
	χ^2 = 60,182, P = 0,000								
Primary pre-service teacher in their 1st year	Subtraction	72	98,6	0	0	1	1,4	73	100
	Parentheses	68	93,2	0	0	5	6,8	73	100
	Fraction	48	65,8	16	21,9	9	12,3	73	100
	Exponentiation	63	86,3	2	2,7	8	11,0	73	100
	Total	251	86,0	18	6,2	23	7,9	292	100
	$\chi^2_{(lr)}$ = 51,788, P = 0,000								
Primary pre-service teacher in their 4th year	Subtraction	94	100	0	0	0	0	94	100
	Parentheses	90	95,7	0	0	4	4,3	94	100
	Fraction	81	86,2	12	12,8	1	1,1	94	100
	Exponentiation	88	93,9	6	6,4	0	0	94	100
	Total	353	93,9	18	4,8	5	1,3	376	100
	$\chi^2_{(lr)}$ = 36,866, P = 0,000								
Future mathematics teachers	Subtraction	42	87,5	0	0	6	12,5	48	100
	Parentheses	39	81,3	0	0	9	18,8	48	100
	Fraction	31	64,6	11	22,9	6	12,5	48	100
	Exponentiation	40	83,3	4	8,3	4	8,3	48	100
	Total	152	79,2	15	7,8	25	13,0	192	100
	$\chi^2_{(lr)}$ = 28,118, P = 0,000								

Table 1 show that the majority of participants represent mathematical operation of subtraction as adequate concept. On the other hand, we can notice that the fraction representations are the least frequently represented as an adequate concept. Subtraction and parentheses are never represented as non-adequate concept. The category "other" is mainly consisted of drawings which suggested that the task was not perceived as a mathematical model-eliciting task; the models presented were more art oriented and did not show mathematical aspects.

If we focus on the differences between groups of participants, we can note that the primary pre-service teachers in their 4th year represent an adequate concept most frequently. We can also see that future mathematics teachers achieve the lowest results at representing the adequate concept.

Below, we used only the drawings which represent adequate concepts of mathematical operations. They were classified in to the categories according to the representation in the drawings. Table 2 show different categorizations which were used in our research.

Table 2. Drawing categorizations formed at mathematical operations

Type of mathematical operation	Categorization of drawing
Subtraction	1. Result symbolically
	2. Result pictorially
	3. Interplay of pictures and symbols
	4. Illustration of numerical expression
	5. Place-value arrangement
	6. Confidential arrangement
Parentheses	1. Result symbolically
	2. Result pictorially
	3. Interplay of pictures and symbols
	4. Illustration of numerical expression
	5. Differentiating summands
	6. Not differentiating summands
Fraction	1. Result symbolically
	2. Result pictorially
	3. Interplay of pictures and symbols
	4. Illustration of numerical expression
	5. Arithmetic model
	6. Geometric model
Exponentiation	1. Result symbolically
	2. Result pictorially
	3. Interplay of pictures and symbols
	4. Illustration of numerical expression
	5. Tree diagram
	6. Nested structure
	7. Cube representation

In the table above, we can see that the categories labelled with numbers 1, 2, 3 and 4 are present in all mathematical operations. With the number 1 (*Result symbolically*), we denote the drawings which include the result of the mathematical expression written with a symbol. With the number 2 (*Result pictorially*), we denote the drawings which include the result of the mathematical expression represented with the picture. With the number 3 (*Interplay of pictures and symbols*), we denote the drawings which include the combination of pictures and symbols. With the term *Symbol*, we denote mathematical symbols (plus, minus, equal …). With the number 4 (*Illustration of numerical expression*), we denote the drawing which includes decorated or illustrated mathematical operation. Other categories were different according to the type of mathematical operation.

At subtraction, we notice two additional categories of drawings (number 5 and 6). For the number 5 (*Place-value arrangement*), we can see that the objects are illustrated in a way, such that the number 17 is shown as 10 in the first line and 7 in the second line. When we subtract the number 9 from the number 17, we do it in the opposite direction. When we cross out these 9 objects from the total number of objects, we can clearly see that only 8 objects are left in the first line. For the number 6 (*Confidential arrangement*), we observe a random distribution of objects, from which we are not able to recognize the total number of objects neither the final result.

At parentheses, we also observe two additional categories (number 5 and 6). For the number 5 (*Differentiating summands*), we can see that the numbers in parentheses in the drawings are presented as two different objects (for example, 4 apples and 5 bananas). For the number 6 (*Not differentiating summands*), the numbers in parentheses in the drawings don't represent two different numbers. They are actually represented as one number (for example, only 9 apples without any bananas).

At fractions, the numbers 5 and 6 represent two types of fraction models. For *arithmetic fraction model*, we can see a set of all objects, which are arranged into the groups with respect to a given fraction expression. For *geometric fraction model*, the fraction is represented as a geometric shape (for example in a shape of a rectangle or a circle).

At exponentiation, the number 5 represents a *tree diagram*, which corresponds to the given exponential expression. Our tree diagram consists of two objects, which are expanded into two additional objects each. Each of these 4 objects are further expanded into two additional objects, which finally gives us 8 objects, which is also the result of our exponential expression. The number 6 represents a *nested structure*, where two tables are presented. On each table, there are two bowls, and each bowl contains two oranges. The number 7 represents a *cube representation*. The cube is presented as geometric object, where the number of vertices is the same the exponent of our exponential expression.

Discussion

We found out that high school students and university students represented mathematical operations of subtraction and parentheses as adequate mathematical concepts. Less abstract concepts (subtraction) are closer to everyday situations, and consequently, the visual representations of those concepts are more common outside of the classroom. On the contrary, exponentiation and fractions are least frequently represented as adequate mathematical concepts. Some other studies also report about teachers having problems with the understanding of fractions (Isik & Kar, 2012; Dixon et al., 2014) and the understanding of exponentiation (Confrey & Smith, 1995). Problems occur at elementary exercises, for example in the following two exercises: »Color ¼ of a given shape (van Steenbrugge et al., 2014; Depaepe et al., 2015)« or »Draw 2^3« (Lipovec & Antolin, 2015).

The knowledge of high school students and university students is closely related to the mathematical knowledge of their teacher (Hill et al., 2005). Therefore, high school students and university students probably perform poorly at representing fractions and exponentiation, because their teachers have a poor understanding of pedagogical content knowledge (Ball et al., 2008). We can conclude from our results that the least abstract mathematical operation – the subtraction is represented quite well, while the most abstract mathematical concept - fractions is represented on weaker level.

Conclusion

The point of graphical representations is actually to help students visualize mathematical concepts and deepen their understanding of those concepts. Mathematics teachers are often not able to recognize the difference between a visualization and a visual representation, which we can not claim for students. Drawing therefore presents an independent mathematical tool for communication, which is quite difficult for interpretation, because a lot of subjective factors of interpreter can influence the result (Dreyfus, 1995).

We found out that high school students and university students can suitably represent some elementary mathematical concepts (subtraction, parentheses, exponentiation and fractions) with a drawing. We also found out that the drawings were more suitable, if they presented less abstract mathematical operations.

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